

CREATION AND ANNIHILATION OPERATORS FOR ORTHOGONAL POLYNOMIALS OF CONTINUOUS AND DISCRETE VARIABLES

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Abstract. We develop general expressions for the raising and lowering operators that belong to the orthogonal polynomials of hypergeometric type with discrete and continuous variable. We construct the creation and annihilation operators that correspond to the normalized polynomials and study their algebraic properties in the case of the Kravchuk/Hermite Meixner/Laguerre polynomials.

1. Introduction. In a previous paper [1] we have developed a method to construct raising and lowering operators for the Kravchuk polynomials of a discrete variable, using the properties of Wigner functions, and to calculate the continuous limit to the creation and annihilation operators for the solutions of the quantum harmonic oscillator.

In this contribution we apply the same method to other orthogonal polynomials of discrete and continuous variable. We give general formulas for all orthogonal polynomials of hypergeometric type [2]: difference/differential equations, recurrence relations, raising and lowering operators.

With the help of standard values we calculate these equations for the normalized functions of Kravchuk-Wigner and Meixner-Laguerre polynomials, and we construct the corresponding creation and annihilation operators.

The motivation of this work is to implement the study of the Sturm-Liouville problem in continuous case with the discrete one, in particular the connexion between the eigenfunctions and the creation and annihilation operators [3] and [4]. This approach is becoming very powerful in the lattice formulation of field theories, where the physical properties of the model are analyzed in the lattice before the continuous limit is taken [5].

2. Basic relations between orthogonal polynomials of continuous and discrete variable. A polynomial of hypergeometric type of continuous variable satisfies the following fundamental equations

1. Differential equation:

$$(C1) \quad \sigma(s) y''_n(s) + \tau(s) y'_n(s) + \lambda_n y_n(s) = 0$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively and λ_n is a constant. This differential equation can be written in the form of an eigenvalue equation

$$(\sigma(s) \rho(s) y'_n(s))' + \lambda_n \rho(s) y_n(s) = 0$$

where $\rho(s)$ is the weight function satisfying $(\sigma(s) \rho(s))' = \tau(s) \rho(s)$ and $\lambda_n = -n(\tau' + \frac{n-1}{2}\sigma'')$.

2. Orthogonality relations:

$$\int_a^b y_n(s) y_m(s) \rho(s) ds = \delta_{nm} d_n^2$$

with d_n some normalization constant,

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3. Three term recurrence relations:

$$(C2) \quad sy_n(s) = \alpha_n y_{n+1}(s) + \beta_n y_n(s) + \gamma_n y_{n-1}(s)$$

where $\alpha_n, \beta_n, \gamma_n$ are constants.

4. Raising and lowering operators:

$$(C3) \quad \sigma(s) y'_n(s) = \frac{\lambda_n}{n\tau'_n} \left[\tau_n(s) y_n(s) - \frac{B_n}{B_{n+1}} y_{n+1}(s) \right]$$

where

$$\begin{aligned} \tau_n(s) &= \tau(s) + n\sigma'(s) \\ \tau'_n &= \tau' + n\sigma'' = -\frac{\lambda_{2n+1}}{2n+1} \end{aligned}$$

We can modify formula (C3) in a more suitable form.

From

$$a_n = B_n \prod_{k=0}^{n-1} \left(\tau' + \frac{1}{2}(n+k-1)\sigma'' \right), \quad a_0 = B_0$$

we can prove the following identity

$$\alpha_n = \frac{a_n}{a_{n+1}} = \frac{B_n}{B_{n+1}} \frac{\left(\tau' + \frac{n-1}{2}\sigma''\right)}{\left(\tau' + \frac{2n-1}{2}\sigma''\right)(\tau' + n\sigma'')} = -\frac{2n(2n+1)}{\lambda_{2n}\lambda_{2n+1}} \frac{\lambda_n}{n} \frac{B_n}{B_{n+1}}$$

that after introducing in (C3) we get a simplified version

$$(C3) \quad \sigma(s) y'_n(s) = -\frac{\lambda_n}{n} \frac{2n+1}{\lambda_{2n+1}} \tau_n(s) y_n(s) - \frac{\lambda_{2n}}{2n} \alpha_n y_{n+1}(s)$$

Finally using the recurrence relations (C2) we get

$$(C4) \quad \sigma(s) y'_n(s) = \left[-\frac{\lambda_n}{n} \frac{2n+1}{\lambda_{2n+1}} \tau_n(s) - \frac{\lambda_{2n}}{2n} (s - \beta_n) \right] y_n(s) + \frac{\lambda_{2n}}{2n} \gamma_n y_{n-1}(s)$$

Formulas (C3) and (C4) can be used to calculate solutions of the differential equations. In fact, we put $n = 0$ in (C4) and we get a differential equation to calculate $y_0(s)$.

Taking this value in (C3) we obtain by iteration all the polynomials satisfying (C1).

We can implement these formulas in the discrete case. A polynomial of hypergeometric type of discrete variable satisfies the following fundamental equations.

1. Difference equation:

$$(D1) \quad \sigma(x) \Delta \nabla y_n(x) + \tau(x) \Delta y_n(x) + \lambda_n y_n(x) = 0$$

where $\sigma(x)$ and $\tau(x)$ are polynomial of at most second and first degree, respectively, and the forward (backward) difference operators are:

$$\Delta f(x) = f(x+1) - f(x) \quad , \quad \nabla f(x) = f(x) - f(x-1)$$

This difference equation can be written in the form of an eigenvalue equation

$$\Delta [\sigma(x) \rho(x) \nabla y_n(x)] + \lambda \rho(x) y_n(x) = 0$$

where $\rho(x)$ is the weight function satisfying

$$\Delta [\sigma(x) \rho(x)] = \tau(x) \rho(x)$$

and

$$\lambda_n = -n\Delta\tau(x) - \frac{n(n-1)}{2}\Delta^2\sigma(x) = -n\left(\tau' + \frac{n-1}{2}\sigma''\right)$$

is the eigenvalue corresponding to the function $y_n(x)$.

2. Orthogonality relations:

The polynomial $P_n(x)$ of hypergeometric type satisfy the following orthogonal relations:

$$\sum_{2\varepsilon=a}^{b-1} P_n(x) P_m(x) \rho(x) = d_n^2 \delta_{mn}$$

where δ_{mn} is the Kronecker symbol and d_n is some normalization constant.

3. Three term recurrence relation:

$$(D2) \quad xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

with $\alpha_n, \beta_n, \gamma_n$ some constants.

4. Raising and lowering operators:

$$(D3) \quad \sigma(x) \nabla P_n(x) = \frac{\lambda_n}{n\tau'_n} \left[\tau_n(s) P_n(x) - \frac{B_n}{B_{n+1}} P_{n+1}(x) \right]$$

where

$$\begin{aligned} \tau_n(s) &= \tau(x+n) + \sigma(x+n) - \sigma(x) \\ \Delta\tau_n(s) &= \Delta\tau(x) + n\Delta^2\sigma(x) \end{aligned}$$

$$\text{or } \tau'_n = \tau' + n\sigma''(x) = -\frac{\lambda_{2n+1}}{2n+1}$$

because $\sigma(x)$ and $\tau(x)$ are functions of at most second and first degree, respectively. We can modify formula (D3) as we did in the continuous case with the help of the identities

$$\begin{aligned} a_n &= B_n \prod_{k=0}^{n-1} \left(\tau' + \frac{1}{2}(n+k-1)\sigma'' \right), a_0 = B_0 \\ \alpha_n &= \frac{a_n}{a_{n+1}} = -\frac{2n}{\lambda_{2n}} \frac{(2n+1)}{\lambda_{2n+1}} \frac{n}{\lambda_n} \frac{B_n}{B_{n+1}} \end{aligned}$$

Introducing these identities in (D3) we get a more simplified version:

$$(D3) \quad \sigma(x) \nabla P_n(x) = -\frac{\lambda_n}{n} \frac{(2n+1)}{\lambda_{2n+1}} \tau_n(x) P_n(x) - \frac{\lambda_{2n}}{2n} \alpha_n P_{n+1}(x)$$

This expression defines the raising operator $P_{n+1}(x)$ in terms of the backward difference operator.

From this expression we can derive another lowering operator in terms of the forward difference operator. We substitute the operator ∇ in (D3) by its equivalent difference operator $\nabla = \Delta - \Delta\nabla$ and then we introduce the difference equation (D1) and the recurrence relation (D2) obtaining

$$(D4) \quad (\sigma(x) + \tau(x)) \Delta P_n(x) = \left[-\frac{\lambda_n}{n} \frac{2n+1}{\lambda_{2n+1}} \tau_n(x) - \lambda_n - \frac{\lambda_{2n}}{2n} x + \frac{\lambda_{2n}}{2n} \beta_n \right] P_n(x) + \\ + \frac{\lambda_{2n}}{2n} \gamma_n P_{n-1}$$

The advantage of expressions (D3) and (D4) is that all the coefficients are tabulated.

As in the continuous case from (D4), putting $n = 0$ we get $P_0(x)$, and inserting this value in (D3) we obtain the solutions of (D1).

From the orthogonal polynomials of hypergeometric type we can construct the corresponding normalized functions (up to a phase factor)

$$\psi_n(x) = d_n^{-1} \sqrt{\rho(x)} P_n(x)$$

that satisfy equivalent relations and we denote them (ND1), (ND2), (ND3), (ND4) in the discrete case and (NC1), (NC2), (NC3), (NC4) in the continuous case. For instance, (NC1) becomes:

$$\sigma(s) \psi_n''(s) + \sigma'(s) \psi_n'(s) - \rho(s)^{-\frac{1}{2}} \left(\sigma(s) \left(\rho(s)^{\frac{1}{2}} \right)' \right)' \psi_n(s) + \lambda_n \psi_n(s) = 0$$

which corresponds to a self-adjoint operator of Sturm-Liouville type.

3. The Hermite and Kravchuk polynomials. We apply the results of section 2 to some orthogonal polynomials of continuous and discrete variable and to the corresponding normalized functions.

For the Hermite polynomials $H_n(s)$ we have

$$(C1) \quad H''_n(s) - 2sH'_n(s) + 2nH(s) = 0$$

$$(C2) \quad sH_n(s) = \frac{1}{2}H_{n+1}(s) + nH_{n-1}(s)$$

$$(C3) \quad H_{n+1}(s) = 2sH_n(s) - H'_n(s)$$

$$(C4) \quad H_{n-1}(s) = \frac{1}{2n}H'_n(s)$$

Introducing the orthonormalized functions

$$\psi_n(s) = (2^n n! \sqrt{\pi})^{-1/2} e^{-s^2/2} H_n(s)$$

we get

$$(NC1) \quad \psi''_n(s) + (2n+1-s^2) \psi_n(s) = 0$$

$$(NC2) \quad 2s\psi_n(s) = \sqrt{2(n+1)}\psi_{n+1}(s) + \sqrt{2n}\psi_{n-1}(s)$$

$$(NC3) \quad \sqrt{n+1}\psi_{n+1}(s) = \frac{1}{\sqrt{2}} \left(s - \frac{d}{ds} \right) \psi_n(s)$$

$$(NC4) \quad \sqrt{n}\psi_{n-1}(s) = \frac{1}{\sqrt{2}} \left(s + \frac{d}{ds} \right) \psi_n(s)$$

(NC1) describes the quantum harmonic oscillator, (NC3) and (NC4) are the realization of the familiar creation and annihilation operator

$$a^+ \equiv \frac{1}{\sqrt{2}} \left(s - \frac{d}{ds} \right) , \quad a \equiv \frac{1}{\sqrt{2}} \left(s + \frac{d}{ds} \right)$$

From (NC4) with $n = 0$ we obtain $\psi_0(s)$ and inserting this value in (NC3) we obtain by iteration the solutions of the harmonic oscillator

$$\psi_n(s) = \frac{1}{\sqrt{n!}} (a^+)^n \psi_0(s)$$

In the discrete case we take the Kravchuk polynomials $k_n^{(p)}(x, N)$ with $x = 0, 1, 2, \dots, N-1$

$$\begin{aligned} \sigma(x) &= x \\ \tau(x) &= (Np - x)/q \quad \tau_n(x) = (Np - x - n)/q + n \\ \lambda_n &= n/q \\ \alpha_n &= n + 1/q \quad \beta_n = n + p(N - 2n), \quad \gamma_n = pq(N - n + 1) \end{aligned}$$

Inserting these values in the fundamental formulas we get

$$(D1) \quad p(N-x)k_n(x+1) + [p(n+x-N) + q(n-x)]k_n(x) + qxk_n(x-1) = 0$$

$$(D2) \quad xk_n(x) = (n+1)k_{n+1}(x) + [n+p(N-2n)]k_n(x) + pq(N-n+1)k_{n-1}(x) = 0$$

$$(D3) \quad (n+1)k_{n+1}(x) = p(x+n-N)k_n(x) + qxk_n(x-1)$$

$$(D4) \quad q(N-n+1)k_{n-1}(x) = (x+n-N)k_n(x) + (N-x)k_n(x+1)$$

For the normalized functions we take the Wigner functions, that appear in the representation of the rotation group, $d_{mm'}^j(\beta)$

$$(-1)^{m-m'} d_{mm'}^j(\beta) = d_n^{-1} \sqrt{\rho(x)} k_n^{(p)}(x, N)$$

$$\text{with } N = 2j \quad m = j - n \quad m' = j - x \quad p = \sin^2 \frac{\beta}{2}, \quad q = \cos^2 \frac{\beta}{2}$$

After substitution we get

$$\begin{aligned} (ND1) \quad &\sqrt{pq(N-x)(x+1)} d_{j-n, j-x-1}^j(\beta) + \\ &+ (p(N-x-n) + q(x-n)) d_{j-n, j-x}^j(\beta) + \\ &+ \sqrt{pqx(N-x+1)} d_{j-n, j-x+1}^j(\beta) = 0 \end{aligned}$$

$$\begin{aligned} (ND2) \quad &[-p(N-x-n) - q(n-x)] d_{j-n, j-x}^j + \\ &+ \sqrt{pq(n+1)(N-n)} d_{j-n-1, j-x}^j(\beta) + \\ &+ \sqrt{pqn(N-n+1)} d_{j-n+1, j-x}^j(\beta) = 0 \end{aligned}$$

$$\begin{aligned} (ND3) \quad &\sqrt{pq(n+1)(N-n)} d_{j-n-1, j-x}^j(\beta) = \\ &= p(N-x-n) d_{j-n, j-x}^j(\beta) + \sqrt{pqx(N-x+1)} d_{j-n, j-x+1}^j \end{aligned}$$

$$\begin{aligned} (ND4) \quad &\sqrt{pqn(N-n+1)} d_{j-n+1, j-x}^j(\beta) = \\ &= p(N-x-n) d_{j-n, j-x}^j(\beta) + \sqrt{pq(x+1)(N-x)} d_{j-n, j-x-1}^j(\beta) \end{aligned}$$

The last four equations can be written down in terms of the new parameters $j = N/2$, $m = j - n$, $m' = j - x$, $p = \sin^2 \beta/2$, $q = \cos^2 \beta/2$, $\sqrt{pq} = \frac{1}{2} \sin \beta$

$$\begin{aligned}
(\text{ND1}) \quad & \sqrt{(j+m')(j-m'+1)} d_{m,m'-1}^j(\beta) + \\
& + \frac{2}{\sin \beta} [m - m' \cos \beta] d_{m,m'}^j(\beta) + \\
& + \sqrt{(j-m')(j+m'+1)} d_{m,m'+1}^j(\beta) = 0 \\
(\text{ND2}) \quad & \sqrt{(j+m)(j-m+1)} d_{m-1,m'}^j(\beta) - \\
& - \frac{2}{\sin \beta} [m' - m \cos \beta] d_{m,m'}^j(\beta) + \\
& + \sqrt{(j-m)(j+m+1)} d_{m+1,m}^j(\beta) = 0 \\
(\text{ND3}) \quad & \frac{1}{2} \sin \beta \sqrt{(j+m)(j-m+1)} d_{m-1,m'}^j(\beta) = \\
& = \sin^2 \frac{\beta}{2} (m+m') d_{m,m'}^j(\beta) + \\
& + \frac{1}{2} \sin \beta \sqrt{(j-m')(j+m'+1)} d_{m,m'+1}^j(\beta) \\
(\text{ND4}) \quad & \frac{1}{2} \sin \beta \sqrt{(j-m)(j+m+1)} d_{m+1,m'}^j(\beta) = \\
& = \sin^2 \frac{\beta}{2} (m+m') d_{m,m'}^j(\beta) + \\
& + \frac{1}{2} \sin \beta \sqrt{(j+m')(j-m'+1)} d_{m,m'-1}^j(\beta)
\end{aligned}$$

Note that (ND1) and (ND2) are equivalent if we interchange $m \leftrightarrow m'$ and take in account the general property of Wigner functions

$$d_{m,m'}^j(\beta) = (-1)^{m-m'} d_{m',m}^j(\beta)$$

The same property of duality applies to (ND3) and (ND4).

In Reference [1] we have constructed creation and annihilation operators with the help of (ND3) and (ND4)

$$A^+ \equiv \frac{1}{\sqrt{2j}} \sqrt{(j+m)(j-m+1)} \quad , \quad A^- \equiv \frac{1}{\sqrt{2j}} \sqrt{(j-m)(j+m+1)}$$

that together with $A^0 \equiv \frac{1}{2j} m$, when apply to the spherical functions Y_{jm} satisfy the $SO(3)$ algebra

$$[A^+, A^-] = 2A^0 \quad , \quad [A^\pm, A^0] = \pm A^\pm.$$

From the connection between the Wigner functions and the solutions of the quantum harmonic oscillator we have proved [1] the limit relations

$$(\text{ND3}) \rightarrow (\text{NC3}) \quad , \quad (\text{ND4}) \rightarrow (\text{NC4})$$

Similar results have been obtained by Bijker et al. [6] for the connection between the $su(2)$ algebra and the one dimensional anharmonic (Morse) oscillator, and by Atakishiev [3] for the lattice implementation of the quantum harmonic oscillator.

3.1. The Laguerre and Meixner polynomials. Using the general formulas of section 2, we get for the Laguerre polynomials $L_n^\alpha(s)$ of continuous variable

- $$\begin{aligned} \text{(C1)} \quad & sL''_n(s) + (1 + \alpha - s)L'_n(s) + nL_n(s) = 0 \\ \text{(C2)} \quad & (n+1)L_{n+1}(s) + (n+s)L_{n-1}(s) = (2n + \alpha + 1 - s)L_n(s) \\ \text{(C3)} \quad & sL'_n(s) = -(1 + \alpha - s)L_n(s) + (n+1)L_{n+1}(s) \\ \text{(C4)} \quad & sL'_n(s) = 2nL_n(s) - (n + \alpha)L_{n-1}(s) \end{aligned}$$

For the normalized Laguerre functions

$$\psi_n(s) = \sqrt{\frac{n!}{\Gamma(\alpha + n + 1)}} e^{-s/2} s^{\alpha/2} L_n^\alpha(s)$$

we obtain the following differential equation, recurrence relations and expressions for the raising and lowering operators:

- $$\begin{aligned} \text{(NC1)} \quad & s^2\psi_n''(s) + s\psi_n'(s) + \frac{1}{2}[-s^2 - \alpha^2 + \alpha s + s]\psi_n(s) + sn\psi_n(s) = 0 \\ \text{(NC2)} \quad & \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) + \sqrt{n(n+\alpha)}\psi_{n-1}(s) = (2n + \alpha + 1 - s)\psi_n^\alpha(s) \\ \text{(NC3)} \quad & \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) = \frac{1}{2}(2n + \alpha + 2 - s)\psi_n(s) + s\psi_n'(s) \\ \text{(NC4)} \quad & \sqrt{n(n+\alpha)}\psi_{n-1}(s) = \frac{1}{2}(2n + \alpha - s) - s\psi_n'(s) \end{aligned}$$

The first equation, divided by s , corresponds to the self-adjoint operator of the Sturm-Liouville problem for the normalized Laguerre functions.

The last two equations can be considered the creation and annihilation operators for the normalized Laguerre function. In fact, from (NC4) with $n=0$ we obtain ψ_0^α . And from (NC3) we easily obtain

$$\psi_n^\alpha(s) = \frac{1}{\sqrt{n!(\alpha+1)_n}} (A^+)^n \psi_0^\alpha$$

where

$$\begin{aligned} A^+\psi_n(s) &= \sqrt{(n+1)(n+\alpha+1)}\psi_{n+1}(s) \\ A^-\psi_n(s) &= \sqrt{n(n+\alpha)}\psi_{n-1}(s) \end{aligned}$$

The Laguerre creation operator was given by Szafraniec [8]. His formula is equivalent to ours if we substitute (NC1) into (NC3).

Now we take the Meixner polynomials $m_n^\gamma(x)$ of discrete variable x , and apply the general expressions of section 2:

- $$\begin{aligned} \text{(D1)} \quad & \mu(x + \gamma)m_n(x + 1) + (x - 1)m_n(x - 1) + [-\mu(x + n + \gamma) + n - x]m_n(x) = 0 \\ \text{(D2)} \quad & \mu m_{n+1}(x) + n(n + \gamma - 1)m_{n-1}(x) = [x(\mu - 1) + n + \mu(n + \gamma)]m_n(x) \\ \text{(D3)} \quad & \mu m_{n+1}(x) = \mu(x + n + \gamma)m_n(x) - xm_n(x - 1) \\ \text{(D4)} \quad & n(n + \gamma - 1)m_{n-1}(x) = \mu(x + n + \gamma)m_n(x) - \mu(\gamma + x)m_n(x + 1) \end{aligned}$$

For the normalized Meixner polynomials

$$M_n^{(\gamma)}(x) = \sqrt{\frac{\mu^n(1-\mu)^\gamma}{n!(\gamma)_n}} \sqrt{\frac{\mu^x \Gamma(\gamma+x)}{\Gamma(x+1) \Gamma(\gamma)}} m_n^\gamma(x)$$

we have the following difference, recurrence equations and raising/lowering operators:

$$\begin{aligned}
 (\text{ND1}) \quad & \sqrt{\mu(\gamma+x)(x+1)}M_n(x+1) + \sqrt{\mu x(x+\gamma-1)}M_n(x-1) - \\
 & - [\mu(x+n+\gamma) - n + x]M_n(x) = 0 \\
 (\text{ND2}) \quad & \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x) + \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x) - \\
 & - [\mu(x+n+\gamma) - x + n]M_n(x) = 0 \\
 (\text{ND3}) \quad & \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x) = \mu(x+n+\gamma)M_n(x) - \\
 & - \sqrt{\mu x(x+\gamma-1)}M_n(x-1) \\
 (\text{ND4}) \quad & \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x) = \mu(x+n+\gamma)M_n(x) - \\
 & - \sqrt{\mu(\gamma+x)(x+1)}M_n(x+1)
 \end{aligned}$$

As in the case of Wigner functions there exists for the Meixner normalized functions an operation of duality between (D1) \leftrightarrow (D2) and (D3) \leftrightarrow (D4) after interchanging $x \leftrightarrow n$.

The last two equations can be consider the creation and annihilation operators for the normalized Meixner functions:

$$\begin{aligned}
 (\text{ND3}) \quad A^+ M_n(x) & \equiv \sqrt{\mu(\gamma+n)(n+1)}M_{n+1}(x) \\
 (\text{ND4}) \quad A^- M_n(x) & \equiv \sqrt{\mu n(n+\gamma-1)}M_{n-1}(x)
 \end{aligned}$$

From $A^- M_0(x) = 0$ and (ND4) we obtain $M_0(x)$ and taking this value into (ND3) we get

$$M_n(x) = \frac{1}{\sqrt{n!(\gamma)_n \mu^n}} (A^+)^n M_0(x)$$

The operators A^+, A^- are adjoint conjugate each other and, together with

$$A^0 = \mu(2n + \gamma)$$

they close the $su(1, 1)$ algebra

$$[A^+, A^-] = A^0 \quad , \quad [A^\pm, A^0] = \pm 2\mu A^\pm$$

Finally, from the limit relation between the Meixner and Laguerre polynomials

$$\frac{1}{n!} m_n^{(\alpha+1, 1-h)} \left(\frac{s}{h} \right) \xrightarrow[h \rightarrow 0]{} L_n^\alpha(s)$$

it can be proved the limit between the corresponding normalized functions

$$M_n^{(\alpha+1, 1-h)} \left(\frac{s}{h} \right) \xrightarrow[h \rightarrow 0]{} \psi_n^\alpha(s)$$

and the limit between the recurrence relations

$$(\text{ND2}) \xrightarrow[h \rightarrow 0]{} (\text{NC2})$$

The Meixner polynomials of discrete variable can be used in the solutions of the three dimensional harmonic oscillator in conection with the energy eigenvalues [Ref. 7, formula (5.10)].

The Laguerre-Meixner creation and annihilation operators were presented by F.H. Szafraniec at Workshop on Orthogonal Polynomials in Mathematical Physics, June 24-26, 1996, Universidad Carlos III de Madrid, Leganes, Spain, (but not published).

Atakishiyev has given recently [10] the hamiltonian and the creation and annihilation operators for the Meixner oscillator. In fact, his formula (43) and (46) are equivalent to ours (ND1), (ND3), (ND4) for the normalized Meixner functions. They satisfy the same $su(1,1)$ algebra if we identify $A^+ \leftrightarrow K_+$, $A^- \leftrightarrow K_-$, $A^0 \leftrightarrow -K_0$

The Laguerre polynomials of continuous variable are used in the solutions of the hydrogen atom, with different weight function in the orthogonality relations [Ref. 7, formula (5.10)].

The wave equation for the hydrogen atom can be considered the differential equation of the Sturm-Liouville problem for the Laguerre functions

$$\psi_n(s) \equiv \psi_n^{2l+1}(s) = \sqrt{\rho_1(s)} L_{n-l-1}^{2l+1}(s)$$

where

$$\rho_1(s) = \sigma(s)\rho(s) = s^{2l+2}e^{-s}$$

In fact, after substituion of $\psi_n(s)$ in (C1) for the Laguerre polynomials we get

$$\psi_n''(s) - [\rho_1(s)]^{-\frac{1}{2}} \left([\rho_1(s)]^{\frac{1}{2}} \right)'' \psi_n(s) + (n - l - 1)s^{-1}\psi_n(s) = 0$$

that corresponds to a self-adjoint operator, from which orthogonality relations for $\psi_n(s)$ with weight function s^{-1} can be obtained.

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REFERENCES

- [1] M. Lorente, "Quantum Mechanics on discrete space and time", in *Fundamental Problems in Quantum Physics* (M. Ferrero, A. van der Merwe, ed.)
- [2] F. Nikiforov, S.K. Suslov, V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer, Berlin 1991.
- [3] N.M. Atakishiev, S.F. Suslov, "Difference analog of the harmonic oscillator", in *Theor. Math. Phys.* 85 (1991) 1055.
- [4] Y.F. Smirnov, "Finite difference equations and factorization method", in *Proceedings of the 5th Wigner Symposium* (P. Kasperkovitz, D. Grau ed.) World Scientific, Singapore 1998.
- [5] M. Lorente, "Induced Representations of the Poincaré group on the lattice: spin 1/2 and 1 case", in *Symmetries in Science X* (B. Gruber, M. Ramek ed.) Plenum Press, New York 1998.
- [6] A. Frank, R. Lemus, R. Bijker, F. Pérez Bernal, J.M. Arias, "A General Algebraic Model for Molecular Vibrational Spectroscopy", *Annals of Phys.* 252 (1996) 211.
- [7] E. Bank, M.E.H. Ismail, "The Attractive Coulomb Potential Polynomials" *Constr. Approx.* (1985) 1:103-119.
- [8] F.H. Szafraniec, "Orthogonal polynomials and subnormality of related shift operators", in *Orthogonal polynomials and their applications*, Proc., Segovia (Spain), 1986, ed. R.C. Palacios, Monograf. Acad. Ci. Zaragoza, vol. 1, pp. 153-155, Zaragoza, 1988.
- [9] N.M. Atakishiyev, E.I. Jafarov, S.M. Nagiyev, K.B. Wolf, "Meixner Oscillators", *Rev. Mex. Fis.* 44 (1988), 235-244.